Integrability of discrete equations modulo a prime

Masataka Kanki
Graduate School of Mathematical Sciences,
University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

Abstract

We apply the 'almost good reduction' (AGR) criterion, which has been introduced in our previous works, to several classes of discrete integrable equations. We first verify our conjecture that AGR can be used as a criterion for integrability of dynamical systems over finite fields, by proving that several q-discrete Painlevé equations have AGR. We then discuss the reduction of a chaotic discrete system and state that AGR is essentially an arithmetic analogue of the singularity confinement method.

Keywords: Integrability test, Good reduction, Discrete Painlevé equation, Finite field.

MSC2010: 37K10, 34M55, 37P25.

1 Introduction

In the theory of arithmetic dynamics, we are interested in how the properties of the mappings change as we make change to the set on which the mappings are defined [1]. In particular, the system over the p-adic integers and its reduction modulo a prime to the finite field attracts much attention. We have another interest to the dynamical systems over finite fields in terms of cellular automata, of which the underlying set consists of a finite number of elements and the mapping is given by recurrence formulae [2]. Mappings are said to have good reductions if, roughly speaking, the reduction and the evolution of the system can commute. One of the typical examples with good reduction is the projective linear group PGL_2. Recently, birational mappings over finite fields have been investigated in terms of integrability [3]. In the previous papers, we defined the generalized notion of good reduction so that it could be applied to wider class of integrable mappings. We called this notion 'almost good reduction' (AGR), and proved that discrete and q-discrete Painlevé II equations have AGR [4, 5]. Our conjecture was that AGR is also satisfied for other discrete Painlevé equations and that AGR is a criterion for integrability of dynamical systems over finite fields. In this paper, we prove that several types of q-discrete analogues of Painlevé equations [6] have AGR for an appropriate domain, thereby verifying the conjecture. We also study the application of AGR to a chaotic system - Hietarinta-Viallet equation [7] - and state that AGR can be seen as an arithmetic analogue of the singularity confinement method [8]. Finally we note on the arithmetic analogue of the algebraic entropy [9, 10].

2 Reduction modulo a prime

Let p be a prime number and for each $x \in \mathbb{Q}$ $(x \neq 0)$ write $x = p^{v_p(x)}\dfrac{u}{v}$ where $v_p(x), u, v \in \mathbb{Z}$ and u and v are coprime integers neither of which is divisible by p. The p-adic norm $|x|_p$ is defined as $|x|_p = p^{-v_p(x)}$. ($|0|_p = 0$.) The local field $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ with respect to the p-adic norm. It is called the field of p-adic numbers and its subring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p | \ |x|_p \leq 1\}$ is called the ring of p-adic integers. The p-adic norm satisfies a non-Archimedean triangle inequality

$$|x+y|_p \leq \max[|x|_p, |y|_p], \tag{1}$$

where equality holds whenever $|x|_p \neq |y|_p$. Let $\mathfrak{p} := p\mathbb{Z}_p$. It is the maximal ideal of $\mathbb{Z}_p$, and can be expressed as

$$\mathfrak{p} = \{x \in \mathbb{Z}_p | \ v_p(x) \geq 1\}.$$

We define $\tilde{x}$ as the reduction of x modulo $\mathfrak{p}$: $\mathbb{Z}_p \ni x \mapsto \tilde{x} \in \mathbb{Z}_p/\mathfrak{p} \cong \mathbb{F}_p$. Since the map $x \mapsto \tilde{x}$ is a ring homomorphism, the following relations satisfy for $x, y \in \mathbb{Z}_p$.

$$\widetilde{x \pm y} = \tilde{x} \pm \tilde{y}, \quad \widetilde{x \cdot y} = \tilde{x} \cdot \tilde{y}, \quad \widetilde{\left(\frac{x}{y}\right)} = \frac{\tilde{x}}{\tilde{y}} \text{ (for } \tilde{y} \neq 0). \tag{2}$$

Let $\phi \in \mathbb{Z}_p(x, y)$ be a rational mapping ϕ: $\mathcal{D} \subseteq \mathbb{Z}_p^2 \to \mathbb{Z}_p^2$ on some domain $\mathcal{D}$, then $\tilde{\phi} \in \mathbb{F}_p(x, y)$ is defined as the mapping with reduced coefficients, The rational map ϕ is said to have a *good reduction* (modulo $\mathfrak{p}$ on the domain $\mathcal{D}$) if we have $\widetilde{\phi(x, y)} = \tilde{\phi}(\tilde{x}, \tilde{y})$ for any $(x, y) \in \mathcal{D}$ [1]. We have generalized this notion so that it can be used to wider class of systems in our previous work:

Definition 1 ([4])
A (non-autonomous) rational map ϕ_n: $\mathcal{D} \subseteq \mathbb{Z}_p^2 \to \mathbb{Q}_p$ $(n \in \mathbb{Z})$ is said to have an almost good reduction modulo $\mathfrak{p}$ if there exists a positive integer $m_{\mathrm{p};n}$ for any $\mathrm{p} = (x, y) \in \mathcal{D}$ and any time step n such that

$$\widetilde{\phi_n^{m_{\mathrm{p};n}}(x, y)} = \widetilde{\phi_n^{m_{\mathrm{p};n}}}(\tilde{x}, \tilde{y}), \tag{3}$$

where $\phi_n^m := \phi_{n+m-1} \circ \phi_{n+m-2} \circ \cdots \circ \phi_n$.

The following map Ψ_γ illustrates how almost good reduction works. Let us define

$$\Psi_\gamma : \begin{cases} x_{n+1} & = \dfrac{a x_n + 1}{x_n^\gamma y_n} \\ y_{n+1} & = x_n \end{cases}, \tag{4}$$

where $|a|_p = 1$ and $\gamma \in \mathbb{Z}_{\geq 0}$ are parameters. The map (4) is known to be integrable if and only if $\gamma = 0, 1, 2$. Let $\mathcal{D}$ be the domain $\{(x, y) \in \mathbb{Z}_p \ |x \neq 0, y \neq 0\}$ then the following proposition holds.

Proposition 1 ([4])
The rational mapping (4) has an almost good reduction modulo $\mathfrak{p}$ if and only if $\gamma = 0, 1, 2$.

We also have proved that the discrete and q-discrete Painlevé II equations, too, have an almost good reduction [4, 5]. Therefore we can postulate that having almost good reduction is equivalent to the integrability of the dynamical systems. In this article, we present further applications of the almost good reduction principle to other integrable equations such as several types of q-discrete Painlevé equations and a chaotic equation.

3 q-difference analogue of Painlevé equations over a finite field

In this section we prove that the q-discrete analogues of Painlevé III, IV and V equations have almost good reductions.

3.1 q-discrete Painlevé III equation

The q-discrete analogue of Painlevé III equation has the following form

$$x_{n+1}x_{n-1} = \frac{ab(x_n - cq^n)(x_n - dq^n)}{(x_n - a)(x_n - b)},$$

where a, b, c, d and q are parameters [6]. It is convenient to rewrite it as the following coupled form

$$\Phi_n : \begin{cases} x_{n+1} &= \dfrac{ab(x_n - cq^n)(x_n - dq^n)}{y_n(x_n - a)(x_n - b)}, \\ y_{n+1} &= x_n. \end{cases} \tag{5}$$

Proposition 2
Suppose that a, b, c, d, q are parameters in $\{1, 2, \cdots, p-1\}$ and that a, b, c, d are distinct and we also suppose that $a+b \not\equiv (c+d)q^3$, then the mapping (5) has an almost good reduction modulo $\mathfrak{p}$ on the domain $\mathcal{D} := \{(x, y) \in \mathbb{Z}_p^2 \,|x \neq a, b,\, y \neq 0\}$.

Proof Let $(x_{n+1}, y_{n+1}) = \Phi_n(x_n, y_n)$. In the case when $\tilde{x}_n \neq \tilde{a}, \tilde{b}$ and $\tilde{y}_n \neq 0$, we have

$$\begin{cases} \tilde{x}_{n+1} &= \dfrac{\tilde{a}\tilde{b}(\tilde{x}_n - \tilde{c}\tilde{q}^n)(\tilde{x}_n - \tilde{d}\tilde{q}^n)}{\tilde{y}_n(\tilde{x}_n - \tilde{a})(\tilde{x}_n - \tilde{b})}, \\ \tilde{y}_{n+1} &= \tilde{x}_n. \end{cases} \tag{6}$$

from the relation (2). Hence $\Phi_n\widetilde{(x_n, y_n)} = \widetilde{\Phi_n}(\tilde{x}_n, \tilde{y}_n)$. We have to examine the other cases. From here we sometimes abbreviate $\tilde{a}$ as a, $\tilde{b}$ as b for simplicity.

(i) If $\tilde{x}_n = \tilde{a}$ and $(a-b)(a+b-cq-dq)\tilde{y}_n^t \not\equiv b(a-c)(a-d)$, neither $\widetilde{\Phi_n}(\tilde{a}, \tilde{y}_n)$ nor $\widetilde{\Phi_n^2}(\tilde{a}, \tilde{y}_n)$ is well-defined. However, $\widetilde{\Phi_n^3}(\tilde{a}, \tilde{y}_n)$ is well-defined and we have,

$$\begin{aligned} \Phi_n^3\widetilde{(x_n, y_n)} &= \widetilde{\Phi_n^3}(\tilde{x}_n = \tilde{a}, \tilde{y}_n) \\ &= \left(\frac{a(b - cq^2)(b - dq^2)\tilde{y}_n}{b(a-c)(a-d) - (a-b)(a+b-cq-dq)\tilde{y}_n}, b\right). \end{aligned}$$

(ii) If $\tilde{x}_n = \tilde{a}$ and $(a-b)(a+b-cq-dq)\tilde{y}_n^t \equiv b(a-c)(a-d)$, none of $\widetilde{\Phi_n^i}(\tilde{a}, \tilde{y}_n)$ is well-defined for $i = 1, 2, 3, 4$. However, $\widetilde{\Phi_n^5}(\tilde{a}, \tilde{y}_n)$ is well-defined and we have,

$$\Phi_n^5 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^5}(\tilde{x}_n = \tilde{a}, \tilde{y}_n) = \left(\frac{b(a-cq^4)(a-dq^4)}{(a-b)(a+b-cq^3-dq^3)}, a \right).$$

(iii) If $\tilde{x}_n = \tilde{b}$ and $(a-b)(a+b-cq-dq)\tilde{y}_n^t \not\equiv -a(b-c)(b-d)$,

$$\begin{aligned} \Phi_n^3 \widetilde{(x_n, y_n)} &= \widetilde{\Phi_n^3}(\tilde{x}_n = \tilde{b}, \tilde{y}_n) \\ &= \left(\frac{b(a-cq^2)(a-dq^2)\tilde{y}_n}{a(b-c)(b-d) + (a-b)(a+b-cq-dq)\tilde{y}_n}, a \right). \end{aligned}$$

(iv) If $\tilde{x}_n = \tilde{b}$ and $(a-b)(a+b-cq-dq)\tilde{y}_n^t \equiv -a(b-c)(b-d)$, we have,

$$\Phi_n^5 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^5}(\tilde{x}_n = \tilde{b}, \tilde{y}_n) = \left(-\frac{a(b-cq^4)(b-dq^4)}{(a-b)(a+b-cq^3-dq^3)}, b \right).$$

(v) If $\tilde{y}_n = 0$ and $\tilde{x}_n \neq 0$,

$$\Phi_n^3 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n, \tilde{y}_n = 0) = \left(0, \frac{ab}{\tilde{x}_n} \right).$$

(vi) If $\tilde{y}_n = 0$ and $\tilde{x}_n = 0$,

$$\Phi_n^4 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^4}(\tilde{x}_n = 0, \tilde{y}_n = 0) = (0, 0).$$

Thus we complete the proof. $\square$

3.2 q-discrete Painlevé IV equation

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{aq^{2n}(x_n^2 + 1) + bq^{2n}x_n}{cx_n + dq^n},$$

where a, b, c, d and q are parameters [6, 11]. It can be rewritten as follows:

$$\Phi_n : \begin{cases} x_{n+1} &= \dfrac{\tau^2(ax_n^2 + bx_n + a) + (x_n y_n - 1)(x_n + \tau)}{x_n(x_n y_n - 1)(x_n + \tau)}, \\ y_{n+1} &= x_n, \end{cases} \tag{7}$$

where $\tau = q^n \tau_0$. Here we took $\tau_0 = d/c$ and redefined a, b as $ac/d^2 \to a$ and $bc/d^2 \to b$.

Proposition 3
Suppose that $|a|_p = |b|_p = |q|_p = |\tau_0|_p = 1$, then the mapping (7) has an almost good reduction modulo $\mathfrak{p}$ on the domain $\mathcal{D} := \{(x, y) \in \mathbb{Z}_p^2 \mid x \neq 0,\ xy \neq 1\ x \neq -q^n \tau_0\ (n \in \mathbb{Z})\}$, if

$$aq^2\tau_0 \neq 1,\ aq^4\tau_0 \neq 1.$$

Proof In the proof we use the abbreviation as $\tilde{a} \to a$, $\tilde{b} \to b, \tilde{\tau}_0 \to \tau_0$.

(i) If $\tilde{x}_n = 0$ and $1 + q^3\tau_0^2 + q^2(-1 - b\tau_0^2 + \tau_0 y_n + a\tau_0 - a\tau_0^2 y_n) \not\equiv 0$,

$$\begin{aligned} &\widetilde{\Phi_n^3(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n = 0, \tilde{y}_n) \\ = &\left(\frac{-1 - q^3\tau_0^2 - bq^4\tau_0^2 + aq^6\tau_0^3 + q^2(1 + b\tau_0^2 - \tau_0\tilde{y}_n + a\tau_0^2\tilde{y}_n)}{q^2\tau_0\{1 + q^3\tau_0^2 + q^2(-1 - b\tau_0^2 + \tau_0\tilde{y}_n + a\tau_0 - a\tau_0^2\tilde{y}_n)\}}, -q^2\tau_0 \right). \end{aligned}$$

(ii) If $\tilde{x}_n = 0$ and $1 + q^3\tau_0^2 + q^2(-1 - b\tau_0^2 + \tau_0 y_n + a\tau_0 - a\tau_0^2 y_n) \equiv 0$,

$$\widetilde{\Phi_n^5(x_n, y_n)} = \widetilde{\Phi_n^5}(\tilde{x}_n = 0, \tilde{y}_n) = \left(\frac{-1 + q^2 + aq^4\tau_0 + q^7\tau_0^2 - bq^8\tau_0^2}{q^4\tau_0(-1 + aq^4\tau_0)}, 0 \right),$$

where we assumed that $aq^4\tau_0 \neq 1$.

(iii) If $\tilde{x}_n = -q^n\tau_0$ and $\tilde{y}_n \neq -\tau_0^{-1}$,

$$\begin{aligned} &\widetilde{\Phi_n^3(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n = -q^n\tau_0, \tilde{y}_n) \\ = &\left(\frac{-1 - \tau_0\tilde{y}_n + (q^3 - bq^4)\tau_0^2(1 + \tau_0\tilde{y}_n) + q^2\{1 + b\tau_0^2 + \tau_0\tilde{y}_n + a\tau_0^2(-\tau_0 + \tilde{y}_n)\}}{q^2\tau_0(-1 + aq^2\tau_0)(1 + \tau_0\tilde{y}_n)}, 0 \right), \end{aligned}$$

where we assumed $aq^2\tau_0 \neq 1$.

(iv) If $\tilde{x}_n = -q^n\tau_0$ and $\tilde{y}_n = -\tau_0^{-1}$,

$$\widetilde{\Phi_n^5(x_n, y_n)} = \widetilde{\Phi_n^5}(\tilde{x}_n = -q^n\tau_0, \tilde{y}_n = -\tau_0^{-1}) = \left(-\frac{1}{aq^6\tau_0^2}, -aq^6\tau_0^2 \right).$$

(v) If $\tilde{x}_n\tilde{y}_n = 1$,

$$\widetilde{\Phi_n^5(x_n, y_n)} = \widetilde{\Phi_n^5}\left(\tilde{x}_n = \frac{1}{\tilde{y}_n}, \tilde{y}_n \right) = \left(\frac{1}{aq^6\tau_0^3\tilde{y}_n}, aq^6\tau_0^3\tilde{y}_n \right).$$

□

3.3 q-discrete Painlevé V equation

The q-discrete Painlevé V equation has a following form:

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{abq^n(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}{(x_n - aq^n)(x_n - bq^n)},$$

where a, b, c, d and q are parameters [6]. It can be rewritten as the following form:

$$\Phi_n : \begin{cases} x_{n+1} &= \dfrac{1}{x_n}\left(\dfrac{abq^n(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}{(x_n - aq^n)(x_n - bq^n)(x_n y_n - 1)} + 1 \right), \\ y_{n+1} &= x_n. \end{cases} \tag{8}$$

Proposition 4

Suppose that a, b, c, d, q are in $\{1, 2, \cdots, p-1\}$ and $a, b, c, d, c^{-1}, d^{-1}$ are distinct from each ohter, then the mapping (8) has almost good reduction modulo $\mathfrak{p}$ on the domain $\mathcal{D} := \{(x, y) \in \mathbb{Z}_p^2 \mid x \neq aq^n, bq^n \ (n \in \mathbb{Z}) \ xy \neq 1\}$.

Proof After lengthy calculations, we obtain the following results:

(i) If $\tilde{x}_n = a$,

$$\Phi_n^3 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n = a, \tilde{y}_n) = \left(\frac{1}{bq}, bq\right).$$

(ii) If $\tilde{x}_n = b$,

$$\Phi_n^3 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n = b, \tilde{y}_n) = \left(\frac{1}{aq}, aq\right).$$

(iii) If $\tilde{x}_n \tilde{y}_n = 1$,

$$\Phi_n^3 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^3}\left(\tilde{x}_n, \tilde{y}_n = \frac{1}{\tilde{x}_n}\right) = \left(\frac{1}{abq\tilde{y}_n}, abq\tilde{y}_n\right).$$

□

3.4 Hietarinta-Viallet equation

The Hietarinta-Viallet equation [7] is the following difference equation:

$$x_{n+1} + x_{n-1} = x_n + \frac{a}{x_n^2}, \tag{9}$$

with a as a parameter. The equation (9) passes the singularity confinement test [8], which is a notable test for integrability of equations, but yet is not integrable in the sense that its algebraic entropy is positive and that the orbits display chaotic behaviours. We prove that the AGR is satisfied for this Hietarinta-Viallet equation. We again rewrite (9) as the following coupled form:

$$\Phi_n : \begin{cases} x_{n+1} &= x_n + \dfrac{a}{x_n^2} - y_n, \\ y_{n+1} &= x_n. \end{cases} \tag{10}$$

Proposition 5
Suppose that $|a|_p = 1$, then the mapping (10) has almost good reduction modulo $\mathfrak{p}$ on the domain $\mathcal{D} := \{(x, y) \in \mathbb{Z}_p^2 \mid x \neq 0\}$.

Proof If $\tilde{x}_n = 0$,

$$\Phi_n^4 \widetilde{(x_n, y_n)} = \widetilde{\Phi_n^4}(\tilde{x}_n = 0, \tilde{y}_n) = (\tilde{y}_n, 0).$$

□

Therefore we learn that the AGR works similary to the singularity confinement test in distinguishing the integrable systems from the non-integrable ones. In fact, the AGR can be seen as an arithmetic analogue of the singularity confinement test.

4 Relation to the 'Diophantine integrability'

Lastly we discuss a relationship between the systems over finite fields and the algebraic entropies of the systems. Let ϕ be a difference equation and let the degree of the map ϕ be $d > 0$. We define the degree of the iterates ϕ^n as $\deg(\phi^n) = d_n$. The naïve composition suggests $d_n = d^n$, however, common factors can be eliminated, lowering the degree of the iterates. Algebraic entropy E of ϕ is the following well-defined quantity [9].

$$E := \lim_{n\to\infty} \frac{1}{n} \log d_n \ (\geq 0)$$

We can postulate from a lot of numerical examples that the mapping ϕ is integrable if and only if $E = 0$, that is, d_n has a polynomial growth.

We can construct an arithmetic analogue of the algebraic entropy which has first been introduced in [10]. If we consider the map with rational numbers as coefficients, and choose initial values to be rational numbers, then we have $x_n \in \mathbb{Q}$ for all $n \in \mathbb{Z}_{>0}$. The arithmetic complexity of rational numbers can be expressed by the height function $H(x)$:

$$H(x) = \max\{|u|, |v|\},$$

where $x = \frac{u}{v}$ and u and v are integers without common factors. ($H(0) = 0$.) The map ϕ is said to be 'Diophantine integrable' if and only if $\log H(x_n)$ grows as slowly as some polynomial. Thus we can define the arithmetic analogue of algebraic entropy as

$$\epsilon := \lim_{n\to\infty} \frac{1}{n} \log\left(\log H(x_n)\right).$$

From the numerical observations we conjecture the followings:

(i) The Hietarinta-Viallet equation (9) has $\epsilon = \log\left(\frac{3+\sqrt{5}}{2}\right)$, which corresponds to the original algebraic entropy $E = \log\left(\frac{3+\sqrt{5}}{2}\right)$ obtained in [7].

(ii) In the case of the equation (4), $\epsilon = \log 3 > 0$ for $\gamma = 3$, while $\epsilon = 0$ for $\gamma = 1, 2$ and $\log H(x_n)$ has a polynomial growth of second degree for generic initial conditions.

This idea is essentially equivalent to studying the growth of the number of digits of the numerator (or denominator) of $x_n \in \mathbb{Q}$ when expressed as p-adic expansions. Therefore the procedure can be seen as an analogue of algebraic entropy of a system over a finite field $\mathbb{F}_p$.

5 Concluding remarks

We studied the integrable discrete equations over a finite field by reducing them from a field of p-adic numbers. We further considered the 'almost good reduction' (AGR), which has been proposed as a criterion for integrability of discrete dynamical systems (over finite fields). We proved that q-discrete Painlevé III, IV and V equations also have AGRs, which has been conjectured in our previous article. We also treated Hietarinta-Viallet equation, which is non-integrable but yet passes singularity confinement test. We proved that it also has an AGR. From these observations we can safely state that the AGR is a criterion for

integrability of discrete dynamics and is also an arithmetic dynamical analogue of the singularity confinement method. Lastly we discussed the arithmetic analogue of the algebraic entropy of the systems. One of the open problems is to modify AGR so that it can determine the integrability of the systems which are non-integrable but yet pass the singularity confinement test, like the Hietarinta-Viallet equation. Other future problems are to investigate the AGR*ness* of the higher dimensional mappings like [12], and to extend our methods to systems with soliton solutions, such as the discrete Korteweg-de Vries equation and the discrete nonlinear Schrödinger equation.

Acknowledgments

The authors wish to thank Professors Jun Mada, K. M. Tamizhmani, Tetsuji Tokihiro and Ralph Willox for insightful discussions and comments. This work is supported by Grant-in-Aid for JSPS Fellows (24-1379).

References

[1] Silverman J H 2007, *The Arithmetic of Dynamical Systems*, Springer-Verlag, New York.

[2] Wolfram S 1981, Statistical mechanics of cellular automata, *Rev. Mod. Phys.* **55**, 601-644.

[3] Roberts J A G 2011, Order and symmetry in birational difference equations and their signatures over finite phase spaces, *Proceedings of the International Workshop Future Directions in Difference Equations*, Vigo, Spain, 213–221.

[4] Kanki M, Mada J, Tamizhmani K M, Tokihiro T 2012 Discrete Painlevé II equation over finite fields, *J. Phys. A: Math. Theor.* **45**, 342001 (8pp).

[5] Kanki M, Mada J, Tamizhmani K M, Tokihiro T 2012 The space of initial conditions and the property of an almost good reduction in discrete Painlevé II equations over finite fields, *preprint*, arXiv:1209.0223.

[6] Ramani A, Grammaticos B and Hietarinta J 1991, Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* **67** 1829–1832.

[7] Hietarinta J, Viallet C 1998 Singularity confinement and chaos in discrete systems, *Phys. Rev. Lett.* **81**, 325–328.

[8] Grammaticos B, Ramani A and Papageorgiou V 1991, Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67**, 1825–1828.

[9] Bellon M P, Viallet C M 1999 Algebraic Entropy, *Comm. Math. Phys.* **204**, 425–437.

[10] Halburd R G 2005 Diophantine Integrability, *J. Phys. A: Math. Gen.* **38**, L263–L269.

[11] Ramani A, Grammaticos B 1996 Discrete Painlevé equations: coalescences, limits and degeneracies, *Physica A* **228**, 160–171.

[12] Kajiwara K, Noumi M and Yamada Y 2002, Discrete dynamical systems with $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry, *Lett. Math. Phys.* **60**, 211–219.